HOT SCALAR THEORY IN LARGE N: BOSE-EINSTEIN CONDENSATION *

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I review the Bose-Einstein condensation phase transition of dilute gases of cold atoms, for particle theorists acquainted with methods of field theory at finite temperature. I then discuss how the dependence of the phase transition temperature on the interaction strength can be computed in the large N approximation.

1 Phase Transitions in Hot Scalar Theories

The standard example from particle physics of a scalar theory is the Higgs sector of electroweak theory, which has a phase transition (or a crossover) at a temperature of order the weak scale: say, a few hundred GeV or so. Let's focus on pure scalar theory by imagining setting the gauge coupling constant $g_{\rm w}$ to zero. At finite temperature, the Higgs picks up a thermal contribution $\#\lambda T^2$ to its effective mass, and the effective potential becomes roughly of the form $V(\phi) \sim m_{\rm eff}^2(T)\,\phi^2 + \lambda\phi^4$ with $m_{\rm eff}^2(T) = -\mathcal{M}^2 + \#\lambda T^2$. At sufficiently high temperature, the $\#\lambda T^2$ turns the potential from being concave down at the origin to concave up, and so restores the symmetry that is spontaneously broken at zero temperature.

Standard techniques for analyzing the phase transition between the hot, symmetry-restored phase and the cold, symmetry-broken phase are as follows. (i) Work in Euclidean time (for studying non-dynamical questions). The Euclidean time direction becomes periodic at finite temperature, with period $\beta = 1/T$. (ii) Near the second-order phase transition ($T \to T_{\rm c}$) of the purely scalar model, the correlation length becomes infinite. Large distance physics becomes important, and there are large, non-perturbative, large-wavelength fluctuations. (iii) At large distances ($E_k \sim k \ll T$), the compact Euclidean time direction decouples, and one can reduce the original Euclidean theory to

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a purely 3-dimensional effective theory of the zero-Euclidean-frequency modes:

$$S_{\rm eff} = \int d^3x \left[|\boldsymbol{\nabla}\phi|^2 + m_{\rm eff}^2 |\phi|^2 + \lambda_{\rm eff} |\phi|^4 + \cdots \right]. \tag{1}$$

(iv) Figure out what to do with the 3-dimensional theory (put it on a lattice, or whatever).

2 Today's Talk: Bose-Einstein Condensation

The purpose of today's talk is to show that the exact same techniques particle physicists use to study relativistically hot scalar theories can also be used to study the Bose-Einstein condensation phase transition of dilute gases of (for example) Rubidium atoms at $T \sim 0.1$ μK. There's an identical three-dimensional effective theory to study the non-perturbative long-distance physics ($E_k \ll T$) near the critical temperature:

$$S_{\rm eff} = \int d^3x \left[|\boldsymbol{\nabla}\phi|^2 + r|\phi|^2 + u|\phi|^4 + \cdots \right], \tag{2}$$

where I've switched to typical condensed-matter names (r and u) for the coefficients. For a non-relativistic problem, $E_k \sim k^2/(2m)$, and the long-distance condition $E_k \ll T$ for the validity of this effective theory becomes $k \ll \sqrt{2mT}$.

The fact that three-dimensional effective theories can describe second-order phase transitions is old hat in condensed matter physics. What's different about the dilute atomic gas problem is that, just as in weakly coupled relativistic problems, the *coefficients* of that effective theory are not phenomenological parameters but can be systematically matched to the microscopic physics! (This allows one to use the effective theory to calculate more than just universal properties of the transition.)

3 Non-Interacting Non-Relativistic Bose Gas

The path integral corresponding to a non-interacting, non-relativistic Bose gas in an external potential $V(x)$ is

$$Z = \int [\mathcal{D}\psi]\, e^{i \int dt\, L}, \tag{3}$$

$$L = \int d^3x\, \psi^* \left(i\partial_t + \frac{1}{2m}\,\nabla^2 - V(x) \right) \psi + (\text{chemical potential term}). \tag{4}$$

where ψ is a complex bosonic field and I've written the path integral, for the moment, in real time rather than Euclidean time. Why is this the path integral? Note that the equation of motion, obtained by varying with respect to ψ^*, is just the Schrödinger equation $\left(i\partial_t + \nabla^2/2m - V(x)\right)\psi = 0$. The above path integral therefore describes the second quantization of the Schrödinger equation: it describes arbitrary numbers of particles, just like the standard path integral for QED describes arbitrary numbers of photons. In canonical quantization language, the field ψ represents an operator

$$\hat{\psi}(\mathbf{x}, t) = \sum_n \hat{a}_n \, \psi_n(\mathbf{x}) \, e^{-i\omega_n t}, \tag{5}$$

where the $\psi_n(\mathbf{x})$ are eigenstates of the Schrödinger equation, the ω_n are the corresponding eigen-energies, and the $\hat{a}_n$ are the corresponding annihilation operators for particles in that mode. If we specialize to the case where there is no external potential $[V(\mathbf{x}) = 0]$, then this becomes

$$\hat{\psi}(\mathbf{x}, t) \to \int_{\mathbf{k}} \hat{a}_{\mathbf{k}} \, e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_k t}, \tag{6}$$

which looks just like the quantization of field in terms of plane waves that you're used to from relativistic quantum field theory.

Recall that in single-particle QM, $\psi^*\psi$ gives you the probability density. In second-quantized QM, the analogous statement is that $\hat{\psi}^*\hat{\psi}$ gives you the number density, so

$$\hat{N} = \int_{\mathbf{x}} \hat{\psi}^*\hat{\psi} = \sum_n \hat{a}_n^\dagger \hat{a}_n. \tag{7}$$

To describe a system of particles with a given number density n, it's convenient to use the grand-canonical ensemble and introduce a chemical potential term μN in the Hamiltonian or Lagrangian. So, our final Lagrangian for a free non-relativistic Bose gas is

$$L = \int d^3x \, \psi^* \left(i\partial_t + \frac{1}{2m}\nabla^2 + \mu - V(x) \right) \psi. \tag{8}$$

4 Interactions

Now let's include a two-body potential $U(\mathbf{x} - \mathbf{y})$ between the atoms (which is adequate in the dilute limit). Remembering that $\psi^*\psi(\mathbf{x})$ is the number density at $\mathbf{x}$, we need

$$L_{\text{int}} = -\frac{1}{2}\int_{\mathbf{xy}} \psi^*\psi(\mathbf{x}) \, U(\mathbf{x} - \mathbf{y}) \, \psi^*\psi(\mathbf{y}). \tag{9}$$

If U is localized, then, in the low-energy limit (wavelength $\gg a$), we can replace it by an effective δ-function:

$$U(\mathbf{x}-\mathbf{y}) \to \frac{4\pi a}{m}\,\delta(\mathbf{x}-\mathbf{y}), \tag{10}$$

where a is the *scattering length* and can be measured for a given type of atom. For this substitution to be valid, the typical inter-particle spacing $n^{-1/3}$ (where n is the number density) should be large compared to the scattering length. In leading non-trivial order in this "diluteness" expansion $a \ll n^{-1/3}$, the Lagrangian for the interacting Bose gas is then simply

$$L = \int d^3x\,\left[\psi^*\left(i\partial_t + \frac{1}{2m}\,\nabla^2 + \mu - V(x)\right)\psi - \frac{2\pi a}{m}(\psi^*\psi)^2\right]. \tag{11}$$

I will henceforth focus on the case of zero external potential V. In that case, this looks just like relativistic ϕ^4 theory, but with $\phi^*(-\partial_t^2)\phi$ replaced by $\psi^* i\partial_t\psi$. In particular, $-\mu$ now plays the role that $m^2_{\rm eff}(T)$ did in the introduction, and one can see that there will be a phase transition depending on whether $-\mu(T) > 0$ (symmetry restored phase) or $-\mu(T) < 0$ (symmetry broken phase). In the latter case, the expectation $\langle\psi\rangle \neq 0$ is what's known as the Bose-Einstein condensate.[a]

5 Now Do the Usual

To study the phase transition, go to Euclidean-time formalism. As in the introduction, note that all but the zero-Euclidean-frequency modes ψ_0 decouple at large distance, leaving

$$\int_0^\beta d\tau\, L_{\rm E} \to \beta\int d^3x\,\left[\psi_0^*\left(-\frac{1}{2m}\,\nabla^2 - \mu_{\rm eff} + V(x)\right)\psi_0 + \frac{2\pi a}{m}(\psi_0^*\psi_0)^2\right]. \tag{12}$$

[Corrections to this effective theory turn out to be higher order in the diluteness expansion.] This Lagrangian can be made to have standard field-theory normalization for the kinetic term by rescaling $\psi_0 = \phi\sqrt{2mT}$:

$$\int_0^\beta d\tau\, L_{\rm E} \to \beta\int d^3x\,\left[|\boldsymbol{\nabla}\phi|^2 + r|\psi|^2 + u|\phi|^4\right], \tag{13}$$

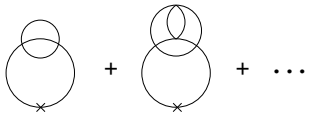
where $r = -2m\mu_{\rm eff}$ and $u = 4\pi amT$. Note that the diluteness condition $a \ll n^{1/3}$ implies that the coupling u should be considered as "small," since

[a] In graduate statistical mechanics, you learned for free particles that the condensate is the particles in the $\mathbf{k} = 0$ mode. The connection is that the contribution of $\mathbf{k} = 0$ to the number density $n = \langle\psi^*\psi\rangle$ is $n_0 = \langle\psi\rangle^*\langle\psi\rangle$.

it is proportional to a. So, we have here an O(2) field theory in 3-dimensions (since ϕ is complex) which is weakly-coupled at short distances.[b]

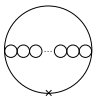
6 A Goal: Calculate T_c as a Function of n

An interesting thing to try with this effective theory is to calculate the correction, due to interactions, to the ideal gas result for the Bose-Einstein condensation temperature T_c. Actually, it turns out to be technially slightly easier to calculate the shift $\Delta n(T)$ in the critical density due to interactions, at fixed temperature, rather than the shift $\Delta T(n)$ in the critical temperature, at fixed density. The two are easily related. Then recall that the density $n = \langle \psi^* \psi \rangle \sim \langle \phi^* \phi \rangle$. If one were to do perturbation theory, the sort of diagrams one would calculate for Δn would then be of the form



where the cross represent the operator $\phi^* \phi$. But perturbation theory breaks down at the transition; so what to do?

One possible technique, implemented by Baym, Blaizot, and Zinn-Justin,[1] is to try the large N approximation for solving the 3-dimensional theory, setting N=2 at the end. At leading order in large N, the graphs which contribute to $\langle \phi^* \phi \rangle \to \langle \vec{\phi} \cdot \vec{\phi} \rangle$ are



Baym *et al.* find $T_c = T_0(1 + 2.33\, a n^{1/3})$ plus higher orders in $1/N$ and in $a n^{1/3}$. Boris Tomášik and I[2] have analyzed the next-order corrections in $1/N$ and find that they change the coefficient 2.33 by only -26% for N=2. This correction is surprisingly small and suggests that large N might not be too bad for T_c!

Acknowledgments

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[b] More technically, this means weakly coupled at the short-distance scale $k \sim \sqrt{2mT}$ where it breaks down, where one wants to match it to the original theory.

References

1. G. Baym, J.-P. Blaizot, and J. Zinn-Justin, *Europhys. Lett.* **49**, 150 (2000).
2. P. Arnold and B. Tomášik, cond-mat/9912306, to appear in *Phys. Rev.* A.